

Load and Resistance Factor Design of Shallow Foundations Against Bearing Failure

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Abstract

Shallow foundation designs are typically governed either by settlement, a serviceability limit state, or by bearing capacity, an ultimate limit state. While geotechnical engineers have been designing against these limit states for over half a century, it is only recently that they have begun to migrate towards reliability-based designs. At the moment, reliability-based design codes are generally derived through calibration with traditional working stress designs. To take advantage of the full potential of reliability-based design the profession must go beyond calibration and take geotechnical uncertainties into account in a rational fashion.

This paper proposes a Load and Resistance Factor Design (LRFD) approach for the bearing capacity design of a strip footing, using load factors as specified by structural codes. The resistance factors required to achieve an acceptable failure probability are estimated as a function of the spatial variability of the soil and by the level of “understanding” of the soil properties in the vicinity of the foundation. The analytical results, validated by simulation, are primarily intended to aid in the development of the next generation of reliability-based geotechnical design codes, but can also be used to assess the reliability of current designs.

Keywords: bearing capacity, reliability, resistance factors, load and resistance factor design, ultimate limit state, shallow foundation

1. Introduction

The design of a shallow footing typically begins with a site investigation aimed at determining the strength and compressibility characteristics of the founding soil or rock. Once this information has been gathered, the geotechnical engineer is in a position to determine the footing dimensions required to avoid entering various limit states. In so doing, it will be assumed here that the geotechnical engineer is in close communication with the structural engineer(s) and is aware of the loads that the footings are being designed to support. The limit states that are usually considered in the footing design are serviceability limit states (typically deformation) and ultimate limit states. The latter is concerned with safety and includes the load-carrying capacity, or *bearing capacity*, of the footing. Figure 1 illustrates a bearing capacity (ultimate limit state) failure of a strip footing founded on a spatially variable soil. The failure surface passes through the lower strength (lighter) regions of the soil.

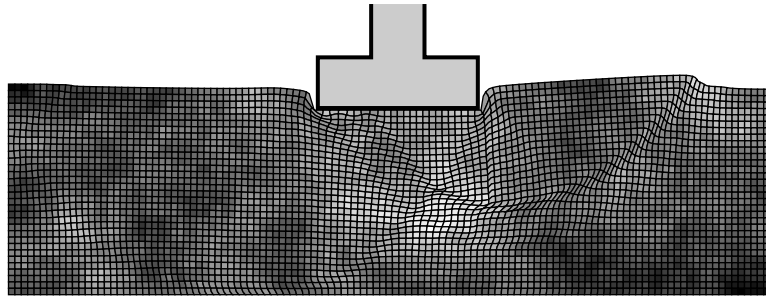


Figure 1. Bearing failure of a strip footing founded on a spatially variable soil by the random finite element method.

This paper develops a load and resistance factor design (LRFD) approach for shallow foundations designed against bearing capacity failure. The design goal is to determine the footing dimensions such that the *ultimate geotechnical resistance* based on characteristic soil properties, \hat{R}_u , satisfies

$$[1] \quad \phi_g \hat{R}_u \geq I \sum_i \alpha_i \hat{L}_i$$

where ϕ_g is the *geotechnical resistance factor*, I is an *importance factor*, α_i is the i 'th *load factor*, and \hat{L}_i is the i 'th *characteristic load effect*. The goal of this paper is to determine the relationship between ϕ_g and the probability that the designed footing will experience a bearing capacity failure. The authors recognize that the use of the symbol ϕ_g is somewhat confusing in a geotechnical paper since the friction angle also uses the symbol ϕ . However, in light of the use of the symbol ϕ for resistance factor by the National Building Code of Canada (NBCC) [National Research Council (NRC), 2005] and other structural codes (see, for example, the use of the symbol ϕ for resistance factor in Commentary K “Foundations” of the User’s Guide – NBC 2005 Structural Commentaries, NRC, 2006). The authors are also adopting the common notation where the subscript denotes the material that the resistance factor governs. For example, where ϕ_c and ϕ_s are resistance factors governing concrete and steel, the letter g in ϕ_g will be taken to denote “geotechnical” or “ground.”

The importance factor, I , reflects the severity of the failure consequences and may be larger than 1.0 for important structures, such as hospitals, whose failure consequences are severe and whose target probabilities of failure are much less than for typical structures. Typical structures usually are

designed using $I = 1$, which will be assumed in this paper. Structures with low failure consequences (minimal risk of loss of life, injury, and/or economic impact) may have $I < 1$.

Only one load combination will be considered in this paper, $\alpha_L \hat{L}_L + \alpha_D \hat{L}_D$, where \hat{L}_L is the characteristic live load, \hat{L}_D is the characteristic dead load, and α_L and α_D are the live and dead load factors, respectively. The load factors used in this paper will be as specified by the National Building Code of Canada (NRC, 2005); $\alpha_L = 1.5$ and $\alpha_D = 1.25$. The theory presented here, however, is easily extended to other load combinations and factors, so long as their (possibly time-dependent) distributions are known.

The characteristic loads will be assumed to be defined in terms of the means of the load components in the following fashion,

$$\begin{aligned} [2a] \quad & \hat{L}_L = k_{L_e} \mu_{L_e} \\ [2b] \quad & \hat{L}_D = k_D \mu_D \end{aligned}$$

where μ_{L_e} and μ_D are the means of the live and dead loads, respectively, and k_{L_e} and k_D are live and dead load *bias* factors, respectively. The bias factors provide some degree of ‘comfort’ by increasing the loads from the mean value to a value having a lesser chance of being exceeded. Since live loads are time varying, the value of μ_{L_e} is more specifically defined as the mean of the maximum live load experienced over a structure’s lifetime (the subscript *e* denotes *extreme*). This definition has the following interpretation: if a series of similar structures, all with the same lifespan, is considered and the maximum live load experienced in each throughout its lifespan is recorded, then a histogram of this set of recorded maximum live loads could be plotted. This histogram then becomes an estimate of the distribution of these extreme live loads and the average of the observed set of maximum values is an estimate of μ_{L_e} . As an aside, the distribution of live load is really quite a bit more complicated than suggested by this explanation, since it actually depends on both spatial position and time (e.g. regions near walls tend to experience much higher live load than seen near the center of rooms). However, historical estimates of live loads are quite appropriately based on spatial averages both conservatively and for simplicity, as discussed next.

For typical multi-story office buildings, Allen (1975) estimates μ_{L_e} to be 1.7 kN/m^2 , based on a 30 year lifetime. The corresponding characteristic live load given by the National Building Code of Canada (NRC, 2005) is $\hat{L}_L = 2.4 \text{ kN/m}^2$, which implies that $k_{L_e} = 2.4/1.7 = 1.41$. Allen further states that the mean live load *at any time* is approximately equal to the 30-year maximum mean averaged over an infinite area. The NBCC provides for a reduction in live loads with tributary area using the formula $0.3 + \sqrt{9.8/A}$, where A is the tributary area ($A > 20 \text{ m}^2$). For $A \rightarrow \infty$, the mean live load at any time is thus approximately $\mu_L = 0.3(1.7) = 0.51 \text{ kN/m}^2$. The bias factor which translates the instantaneous mean live load, μ_L to the characteristic live load, \hat{L}_L , is thus quite large having value $k_L = 2.4/0.51 = 4.7$.

Dead load, on the other hand, is largely static, and the time span considered (e.g. lifetime) has little effect on its distribution. Becker (1996) estimates k_D to be 1.18. Figure 2 illustrates typical probability density functions for the three types of loads (instantaneous live, extreme live, and dead) commonly considered and the relative locations of the characteristic load values.

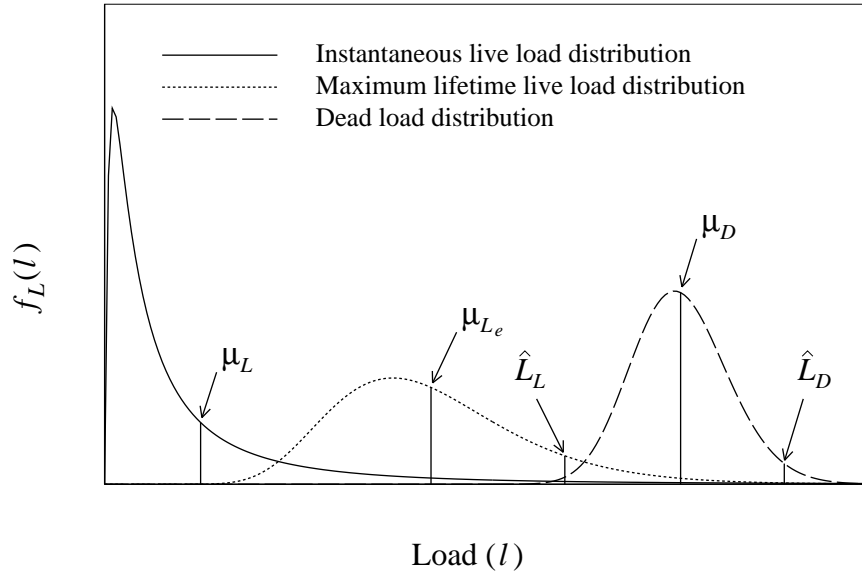


Figure 2. Characteristic and mean values of live and dead loads along with their corresponding distributions.

The characteristic ultimate geotechnical resistance, \hat{R}_u , is determined using characteristic soil properties, in this case characteristic values of the soil's cohesion, c , and friction angle, ϕ (note that although the primes are omitted from these quantities it should be recognized that the theoretical developments described in this paper are applicable to either total or effective strength parameters). To obtain the characteristic soil properties, the soil is sampled over a single column somewhere in the vicinity of the footing, for example, a single CPT or SPT sounding near the footing. The sample is assumed to yield a sequence of m observed cohesion values, $c_1^o, c_2^o, \dots, c_m^o$, and m observed friction angle values, $\phi_1^o, \phi_2^o, \dots, \phi_m^o$. The superscript 'o' denotes an observation. It is assumed here that the observations are error-free, which is an *unconservative* assumption. If the actual observations have considerable error, then the resistance factor used in the design should be reduced. This issue is discussed further in the conclusions.

The characteristic value of the cohesion, \hat{c} , is defined in this paper as the median of the sampled observations, c_i^o , which, assuming c is lognormally distributed, can be computed using the geometric average,

$$[3] \quad \hat{c} = \left[\prod_{i=1}^m c_i^o \right]^{1/m} = \exp \left\{ \frac{1}{m} \sum_{i=1}^m \ln c_i^o \right\}$$

The geometric average is used here because if c is lognormally distributed, as assumed, then \hat{c} will also be lognormally distributed.

The characteristic value of the friction angle is computed as an arithmetic average

$$[4] \quad \hat{\phi} = \frac{1}{m} \sum_{i=1}^m \phi_i^o$$

The arithmetic average is used here because ϕ is assumed to follow a symmetric bounded distribution and the arithmetic average preserves the mean. That is, the mean of $\hat{\phi}$ is the same as the mean of ϕ .

To determine the characteristic ultimate geotechnical resistance, \hat{R}_u , it will first be assumed that the soil is weightless. This simplifies the calculation of the ultimate bearing stress, q_u , to

$$[5] \quad q_u = cN_c$$

The assumption of weightlessness is conservative since the soil weight contributes to the overall bearing capacity. This assumption also allows the analysis to explicitly concentrate on the role of cN_c on ultimate bearing capacity, since this is the only term that includes the effects of spatial variability relating to *both* shear strength parameters c and ϕ .

Bearing capacity predictions, involving specification of the N_c factor in this case, are generally based on plasticity theories (see, e.g., Prandtl, 1921, Terzaghi, 1943, and Sokolovski, 1965) in which a rigid base is punched into a softer material. These theories assume that the soil underlying the footing has properties which are spatially constant (everywhere the same). This type of ideal soil will be referred to as a *uniform soil* henceforth. Under this assumption, most bearing capacity theories (e.g., Prandtl, 1921, and Meyerhof, 1951, 1963) assume that the failure slip surface takes on a logarithmic spiral shape to give

$$[6] \quad N_c = \frac{e^{\pi \tan \phi} \tan^2 \left(\frac{\pi}{4} + \frac{\phi}{2} \right) - 1}{\tan \phi}$$

See Griffiths and Fenton (2001) for a probabilistic analysis of a footing founded on a purely cohesive ($\phi = 0$) soil. The current paper considers the more general case of a $c - \phi$ soil. One can always set $\phi = 0$ in the following theory to perform a total stress analysis on an “undrained clay” soil.

Consistent with the theoretical results presented by Fenton et al. (2007), this paper will also concentrate on the design of a strip footing, as illustrated in Figure 1. In this case, the characteristic ultimate geotechnical resistance, \hat{R}_u , becomes

$$[7] \quad \hat{R}_u = B\hat{q}_u$$

where B is the footing width and \hat{R}_u has units of load per unit length out-of-plane, i.e., in the direction of the strip foot. The characteristic ultimate bearing stress, \hat{q}_u , is defined by

$$[8] \quad \hat{q}_u = \hat{c}\hat{N}_c$$

where the characteristic N_c factor is determined using the characteristic friction angle in Eq. 6,

$$[9] \quad \hat{N}_c = \frac{e^{\pi \tan \hat{\phi}} \tan^2 \left(\frac{\pi}{4} + \frac{\hat{\phi}}{2} \right) - 1}{\tan \hat{\phi}}$$

For the strip footing and just the dead and live load combination, the load and resistance factor design (LRFD) equation becomes

$$[10] \quad \phi_g B \hat{q}_u = I [\alpha_L \hat{L}_L + \alpha_D \hat{L}_D] \quad \Rightarrow \quad B = \frac{I [\alpha_L \hat{L}_L + \alpha_D \hat{L}_D]}{\phi_g \hat{q}_u}$$

To determine the resistance factor, ϕ_g , required to achieve a certain acceptable reliability of the constructed footing, it is necessary to estimate the probability of bearing capacity failure of a

footing designed using Eq. 10. This paper will use the theoretical results presented by Fenton et al. (2007) for a strip footing, which will be summarized in Section 3. Once the probability of failure, p_f , for a certain design using a specific value for ϕ_g , is known this probability can be compared to the maximum acceptable failure probability, p_m . If p_f exceeds p_m , then the resistance factor must be reduced and the footing re-designed. Similarly, if p_f is less than p_m , then the design is over-conservative and the value of ϕ_g can be increased. A specific relationship between p_m and ϕ_g will be given here. Design curves will also be presented from which the value of ϕ_g required to achieve a maximum acceptable failure probability can be determined.

As suggested above, the determination of the required resistance factor, ϕ_g , involves deciding on a maximum acceptable failure probability, p_m . The choice of p_m derives from a consideration of acceptable risk and directly influences the size of ϕ_g . Different levels of p_m may be considered to reflect the “importance” of the supported structure – p_m may be much smaller for a hospital than for an uninhabited storage warehouse.

The choice of a maximum failure probability, p_m , should consider the margin of safety implicit in current foundation designs and the levels of reliability for geotechnical design as reported in the literature. The values of p_m for foundation designs are nearly the same or somewhat less than those for concrete and steel structures because of the difficulties and high expense of foundation repairs. A literature review of the suggested acceptable probability of failure for foundations is listed in Table 1.

Table 1. Literature review of lifetime probabilities of failure of foundations.

Source	p_m
Meyerhof (1970, 1993, 1995)	$10^{-2} - 10^{-4}$
Simpson et al. (1981)	10^{-3}
NCHRP (1991)	$10^{-2} - 10^{-4}$
Becker (1996)	$10^{-3} - 10^{-4}$

Meyerhof (1995) was quite specific about acceptable risks: “The order of magnitude of lifetime probabilities of stability failure is about 10^{-2} for offshore foundation, about 10^{-3} for earthworks and earth retaining structures, and about 10^{-4} for foundations on land.”

In this paper three maximum lifetime failure probabilities, 10^{-2} , 10^{-3} , and 10^{-4} will be considered. In general, and without regard to the structural categorizations made by Meyerhof above, these probabilities are deemed by the authors to be appropriate for designs involving low, medium and high failure consequence structures, respectively. Resistance factors required to achieve these target probabilities will be recommended for the specific $c - \phi$ soil considered. These resistance factors are smaller than those the theory suggests for an undrained soil, since a $\phi = 0$ soil has only one source of uncertainty. In other words, the resistance factors based on a generalized $c - \phi$ soil are considered to be reasonably conservative.

It is also noted that the effect of structural importance should actually be reflected in the importance factor, I , of Eq. 1 and not in the resistance factor. The resistance factor should be aimed at a medium, or common, structural importance level and the importance factor should be varied above and below 1.0 to account for more and less important structures, respectively. However, since acceptable failure probabilities may not be simply connected to structural importance, $I = 1$ is assumed in the following. For code provisions, the factors recommended here should be considered to be the ratio ϕ_g/I

2. The random soil model

The soil cohesion, c , is assumed to be lognormally distributed with mean μ_c , standard deviation σ_c , and spatial correlation length $\theta_{\ln c}$. A lognormally distributed random field is obtained from a normally distributed random field, $G_{\ln c}(\underline{x})$, having zero mean, unit variance, and spatial correlation length $\theta_{\ln c}$ through the transformation

$$[11] \quad c(\underline{x}) = \exp\{\mu_{\ln c} + \sigma_{\ln c} G_{\ln c}(\underline{x})\}$$

where \underline{x} is the spatial position at which c is desired, $\sigma_{\ln c}^2 = \ln(1 + v_c^2)$, $\mu_{\ln c} = \ln(\mu_c) - \sigma_{\ln c}^2/2$, and $v_c = \sigma_c/\mu_c$ is the coefficient of variation.

The correlation coefficient between the log-cohesion at a point \underline{x}_1 and a second point \underline{x}_2 is specified by a correlation function, $\rho_{\ln c}(\underline{\tau})$, where $\underline{\tau} = \underline{x}_1 - \underline{x}_2$ is the vector between the two points. In this paper, a simple exponentially decaying (Markovian) correlation function will be assumed, having the form

$$[12] \quad \rho_{\ln c}(\underline{\tau}) = \exp\left(-\frac{2|\underline{\tau}|}{\theta_{\ln c}}\right)$$

where $|\underline{\tau}| = \sqrt{\tau_1^2 + \tau_2^2}$ is the length of the vector τ . The spatial correlation length, $\theta_{\ln c}$, is loosely defined as the separation distance within which two values of $\ln c$ are significantly correlated. Mathematically, $\theta_{\ln c}$ is defined as the area under the correlation function, $\rho_{\ln c}(\underline{\tau})$ (Vanmarcke, 1984).

The spatial correlation function, $\rho_{\ln c}(\underline{\tau})$ has a corresponding variance reduction function, $\gamma_{\ln c}(D)$, which specifies how the variance is reduced upon local averaging of $\ln c$ over some domain D . In the two dimensional analysis considered here, $D = D_1 \times D_2$ is an area and the two-dimensional variance reduction function is defined by

$$[13] \quad \gamma_{\ln c}(D_1, D_2) = \frac{4}{(D_1 D_2)^2} \int_0^{D_1} \int_0^{D_2} (D_1 - \tau_1)(D_2 - \tau_2) \rho(\tau_1, \tau_2) d\tau_1 d\tau_2$$

which can be evaluated using Gaussian quadrature (see Fenton and Griffiths, 2003, and Griffiths and Smith, 2006, for more details).

It should be emphasized that the correlation function selected above acts between values of $\ln c$. This is because $\ln c$ is normally distributed and a normally distributed random field is simply defined by its mean and covariance structure. In practice, the correlation length $\theta_{\ln c}$ can be estimated by evaluating spatial statistics of the log-cohesion data directly (see, e.g., Fenton, 1999). Unfortunately, such studies are scarce so that little is currently known about the spatial correlation structure of natural soils. For the problem considered here, it turns out that a worst case correlation length exists which can be conservatively assumed in the absence of improved information.

The random field is also assumed here to be statistically isotropic (the same correlation length in any direction through the soil). Although the horizontal correlation length is often greater than the vertical, due to soil layering, taking this into account was deemed to be a site specific refinement which does not lead to an increase in the general understanding of the probabilistic behaviour of shallow foundations. The theoretical results presented here, however, apply also to anisotropic soils, so that the results are easily extended to specific sites. The authors have found that when the

soil is sampled at some distance from the footing (i.e. not directly under the footing) that increasing the correlation length in the horizontal direction to values above the worst-case isotropic correlation length leads to a decreased failure probability, so that the isotropic case is also conservative for low to medium levels of site understanding. When the soil is sampled directly below the footing, the failure probability increases as the horizontal correlation length is increased above the worst case scale, which is unconservative.

The friction angle, ϕ , is assumed to be bounded both above and below, so that neither normal nor lognormal distributions are appropriate. A beta distribution is often used for bounded random variables. Unfortunately, a beta distributed random field has a complex joint distribution and simulation is cumbersome and numerically difficult. To keep things simple, a bounded distribution is selected which resembles a beta distribution but which arises as a simple transformation of a standard normal random field, $G_\phi(\underline{x})$, according to

$$[14] \quad \phi(\underline{x}) = \phi_{min} + \frac{1}{2}(\phi_{max} - \phi_{min}) \left\{ 1 + \tanh \left(\frac{sG_\phi(\underline{x})}{2\pi} \right) \right\}$$

where ϕ_{min} and ϕ_{max} are the minimum and maximum friction angles in radians, respectively, and s is a scale factor which governs the friction angle variability between its two bounds (see Fenton and Griffiths, 2008, for more details). Figure 3 shows how the distribution of ϕ (normalized to the interval $[0, 1]$) changes as s changes, going from an almost uniform distribution at $s = 5$ to a very normal looking distribution for smaller s . Thus, varying s between about 0.1 and 5.0 leads to a wide range in the stochastic behaviour of ϕ . In all cases, the distribution is assumed to be symmetric so that the midpoint between ϕ_{min} and ϕ_{max} is the mean. Values of s less than 1 lead to narrower distributions, however those with s greater than about 5 lead to a U-shaped distributions (higher at the boundaries), which are unrealistic.

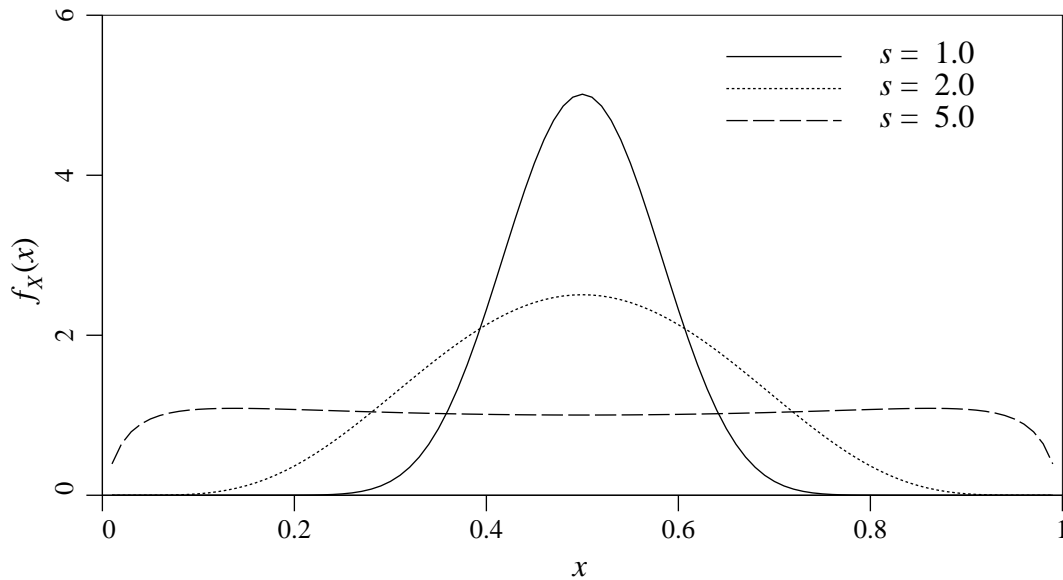


Figure 3. Bounded distribution of friction angle normalized to the interval $[0, 1]$.

The following relationship between s and the variance of ϕ derives from a third-order Taylor series approximation to \tanh and a first-order approximation to the final expectation,

$$\begin{aligned}
 [15] \quad \sigma_\phi^2 &= (0.5)^2(\phi_{max} - \phi_{min})^2 \mathbf{E} \left[\tanh^2 \left(\frac{sG_\phi}{2\pi} \right) \right] \\
 &\simeq (0.5)^2(\phi_{max} - \phi_{min})^2 \mathbf{E} \left[\frac{\left(\frac{sG_\phi}{2\pi} \right)^2}{1 + \left(\frac{sG_\phi}{2\pi} \right)^2} \right] \\
 &\simeq (0.5)^2(\phi_{max} - \phi_{min})^2 \frac{s^2}{4\pi^2 + s^2}
 \end{aligned}$$

where $\mathbf{E}[\dots]$ is the expectation operator and $\mathbf{E}[G_\phi^2] = 1$ since G_ϕ is a standard normal random variable. Eq. 15 slightly overestimates the true standard deviation of ϕ , from 0% when $s = 0$ to 11% when $s = 5$. A closer approximation over the entire range $0 \leq s \leq 5$ is obtained by slightly decreasing the 0.5 factor to 0.46 (this is an empirical adjustment),

$$[16] \quad \sigma_\phi \simeq \frac{0.46(\phi_{max} - \phi_{min})s}{\sqrt{4\pi^2 + s^2}}$$

The close agreement is illustrated in Figure 4.

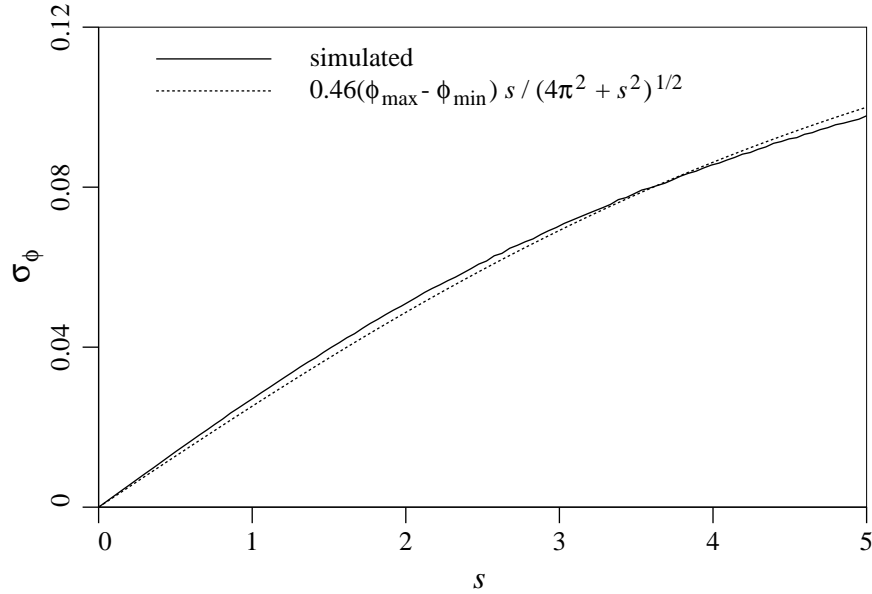


Figure 4. Relationship between σ_ϕ and s derived from simulation (100,000 realizations for each s) and the Taylor's series derived approximation given by Eq. 16. The vertical scale corresponds to $\phi_{max} - \phi_{min} = 0.349$ radians (20 degrees).

Eq. 15 can be generalized to yield the covariance between $\phi(\mathbf{x}_i)$ and $\phi(\mathbf{x}_j)$, for any two spatial points \mathbf{x}_i and \mathbf{x}_j as follows,

$$[17] \quad \text{Cov} [\phi(\mathbf{x}_i), \phi(\mathbf{x}_j)] = (0.5)^2(\phi_{max} - \phi_{min})^2 \mathbf{E} \left[\tanh \left(\frac{sG_\phi(\mathbf{x}_i)}{2\pi} \right) \tanh \left(\frac{sG_\phi(\mathbf{x}_j)}{2\pi} \right) \right]$$

$$\begin{aligned}
&\simeq (0.5)^2(\phi_{max} - \phi_{min})^2 \mathbf{E} \left[\frac{\left(\frac{sG_\phi(\mathcal{X}_i)}{2\pi} \right) \left(\frac{sG_\phi(\mathcal{X}_j)}{2\pi} \right)}{1 + \frac{1}{2} \left(\left(\frac{sG_\phi(\mathcal{X}_i)}{2\pi} \right)^2 + \left(\frac{sG_\phi(\mathcal{X}_j)}{2\pi} \right)^2 \right)} \right] \\
&\simeq (0.46)^2(\phi_{max} - \phi_{min})^2 \frac{s^2 \rho_\phi(\mathcal{X}_i - \mathcal{X}_j)}{4\pi^2 + s^2} \\
&= \sigma_\phi^2 \rho_\phi(\mathcal{X}_i - \mathcal{X}_j)
\end{aligned}$$

where the empirical correction found in Eq. 16 was introduced in the second last step.

It seems reasonable to assume that if the spatial correlation structure of a soil is caused by changes in the constitutive nature of the soil over space, then both cohesion and friction angle would have similar correlation lengths. Thus, θ_ϕ is taken to be equal to $\theta_{\ln c}$ in this study and ϕ is assumed to have the same correlation structure as c (Eq. 12), i.e., $\rho_\phi(\tau) = \rho_{\ln c}(\tau)$. Both correlation lengths will be referred to generically from now on simply as θ , and both correlation functions as $\rho(\tau)$, remembering that this length and correlation function reflects correlation between points in the underlying normally distributed random fields, $G_{\ln c}(x)$ and $G_\phi(x)$, and not directly between points in the cohesion and friction fields (although the correlation lengths in the different spaces are quite similar). The correlation lengths can be estimated by statistically analyzing data generated by inverting Eq's 11 and 14. Since both fields have the same correlation function, $\rho(\tau)$, they will also have the same variance reduction function, i.e., $\gamma_{\ln c}(D) = \gamma_\phi(D) = \gamma(D)$, as defined by Eq. 13.

The two random fields, c and ϕ , are assumed to be independent. Non-zero correlations between c and ϕ were found by Fenton and Griffiths (2003) to have only a minor influence on the estimated probabilities of bearing capacity failure. Since the general consensus is that c and ϕ are negatively correlated (Cherubini, 2000, and Wolff, 1985) and the mean bearing capacity for independent c and ϕ was slightly lower than for the negatively correlated case (Fenton and Griffiths, 2003), the assumption of independence between c and ϕ is slightly conservative.

3. Analytical approximation to the probability of failure

In this section, an analytical approximation to the probability of bearing capacity failure of a strip footing is summarized. Eq. 5 was developed assuming an ideal soil whose shear strength is everywhere the same (i.e. a *uniform soil*). When soil properties are spatially variable, as they are in reality, then the hypothesis made in this study is that Eq. 5 can be replaced by

$$[18] \quad q_u = \bar{c} \bar{N}_c$$

where \bar{c} and \bar{N}_c are the *equivalent* cohesion and *equivalent* N_c factor, defined as those *uniform* soil parameters which lead to the same bearing capacity as observed in the real, spatially varying, soil. In other words, it is proposed that equivalent soil properties, \bar{c} and $\bar{\phi}$, exist such that a uniform soil having these properties will have the same bearing capacity as the actual spatially variable soil. The value of \bar{N}_c is obtained by using the equivalent friction angle, $\bar{\phi}$, in Eq. 6,

$$[19] \quad \bar{N}_c = \frac{e^{\pi \tan \bar{\phi}} \tan^2 \left(\frac{\pi}{4} + \frac{\bar{\phi}}{2} \right) - 1}{\tan \bar{\phi}}$$

In the design process, Eq. 18 is replaced by Eq. 8 and the design footing width, B , is obtained using Eq. 10, which, in terms of the characteristic design values becomes

$$[20] \quad B = \frac{I [\alpha_L \hat{L}_L + \alpha_D \hat{L}_D]}{\phi_g \hat{c} \hat{N}_c}$$

The design philosophy proceeds as follows: find the required footing width B such that the probability that the actual load, L , exceeds the actual resistance, $q_u B$, is less than some small acceptable failure probability, p_m . If p_f is the actual failure probability, then

$$[21] \quad p_f = \mathbf{P}[L > q_u B] = \mathbf{P}[L > \bar{c} \bar{N}_c B]$$

and a successful design methodology will have $p_f \leq p_m$. Substituting Eq. 20 into Eq. 21 and collecting random terms to the left of the inequality leads to

$$[22] \quad p_f = \mathbf{P} \left[L \frac{\hat{c} \hat{N}_c}{\bar{c} \bar{N}_c} > \frac{I [\alpha_L \hat{L}_L + \alpha_D \hat{L}_D]}{\phi_g} \right]$$

Letting

$$[23] \quad Y = L \frac{\hat{c} \hat{N}_c}{\bar{c} \bar{N}_c}$$

means that

$$[24] \quad p_f = \mathbf{P} \left[Y > \frac{I [\alpha_L \hat{L}_L + \alpha_D \hat{L}_D]}{\phi_g} \right]$$

and the task is to find the distribution of Y . Assuming that Y is lognormally distributed (an assumption found to be reasonable by Fenton et al., 2007, and which is also supported to some extent by the central limit theorem), then

$$[25] \quad \ln Y = \ln L + \ln \hat{c} + \ln \hat{N}_c - \ln \bar{c} - \ln \bar{N}_c$$

is normally distributed and p_f can be found once the mean and variance of $\ln Y$ are determined. The mean of $\ln Y$ is

$$[26] \quad \mu_{\ln Y} = \mu_{\ln L} + \mu_{\ln \hat{c}} + \mu_{\ln \hat{N}_c} - \mu_{\ln \bar{c}} - \mu_{\ln \bar{N}_c}$$

and the variance of $\ln Y$ is

$$[27] \quad \sigma_{\ln Y}^2 = \sigma_{\ln L}^2 + \sigma_{\ln \hat{c}}^2 + \sigma_{\ln \bar{c}}^2 + \sigma_{\ln \hat{N}_c}^2 + \sigma_{\ln \bar{N}_c}^2 - 2\text{Cov}[\ln \bar{c}, \ln \hat{c}] - 2\text{Cov}[\ln \bar{N}_c, \ln \hat{N}_c]$$

where the load, L , and soil properties, c and ϕ have been assumed mutually independent.

To find the parameters in Eq's 26 and 27, the following two assumptions are made;

- 1) the equivalent cohesion, \bar{c} , is the geometric average of the cohesion field over some zone of influence, D , under the footing,

$$[28] \quad \bar{c} = \exp \left\{ \frac{1}{D} \int_D \ln c(\underline{x}) d\underline{x} \right\}$$

Note that in this two-dimensional analysis, D is an area and the above is a two-dimensional integration. If $c(\underline{x})$ is lognormally distributed, as assumed, then \bar{c} is also lognormally distributed.

- 2) the equivalent friction angle, $\bar{\phi}$, is the arithmetic average of the friction angle over the zone of influence, D ,

$$[29] \quad \bar{\phi} = \frac{1}{D} \int_D \phi(\underline{x}) d\underline{x}$$

This relationship also preserves the mean, i.e. $\mu_{\bar{\phi}} = \mu_{\phi}$.

Probably the greatest source of uncertainty in this analysis involves the choice of the domain, D , over which the equivalent soil properties are averaged under the footing. The averaging domain was found by trial and error to be best approximated by $D = W \times W$, centered directly under the footing (see Figure 5). In this study, W is taken as 80% of the average mean depth of the wedge zone directly beneath the footing, as given by the classical Prandtl failure mechanism,

$$[30] \quad W = \frac{0.8}{2} \hat{\mu}_B \tan \left(\frac{\pi}{4} + \frac{\mu_{\phi}}{2} \right)$$

and where μ_{ϕ} is the mean friction angle (in radians), within the zone of influence of the footing, and $\hat{\mu}_B$ is an estimate of the mean footing width obtained by using mean soil properties (μ_c and μ_{ϕ}) in Eq. 10,

$$[31] \quad \hat{\mu}_B = \frac{I [\alpha_L \hat{L}_L + \alpha_D \hat{L}_D]}{\phi_g \mu_c \mu_{N_c}}$$

The footing shown on Figure 5 is just one possible realization since the footing width, B , is actually a random variable. The averaging area D with dimension W suggested by Eq. 30 is significantly smaller than that suggested by Fenton and Griffiths (2003). The 2003 study assumed that the footing width was known, rather than designed, and recognized that the larger averaging region did not well represent the mean bearing capacity, which of course is the most important value in probability calculations. The smaller averaging region used in this study may be reasonable if one considers the actual quantity of soils involved in resisting the bearing failure along the failure surfaces. That is, D would be the area of soil which deforms during failure. Since this area will change, sometimes dramatically, from realization to realization, the above can only be considered a rough empirical approximation. The problem of deciding on an appropriate averaging region needs further study. In the simulations performed to validate the theory presented here, the soil depth is taken to be $H = 4.8$ m and $\Delta x = 0.15$ m, where Δx is the width of the columns of finite elements used in the simulations (see Figure 1).

To first order, the mean of N_c is,

$$[32] \quad \mu_{N_c} \simeq \frac{e^{\pi \tan \mu_{\phi}} \tan^2 \left(\frac{\pi}{4} + \frac{\mu_{\phi}}{2} \right) - 1}{\tan \mu_{\phi}}$$

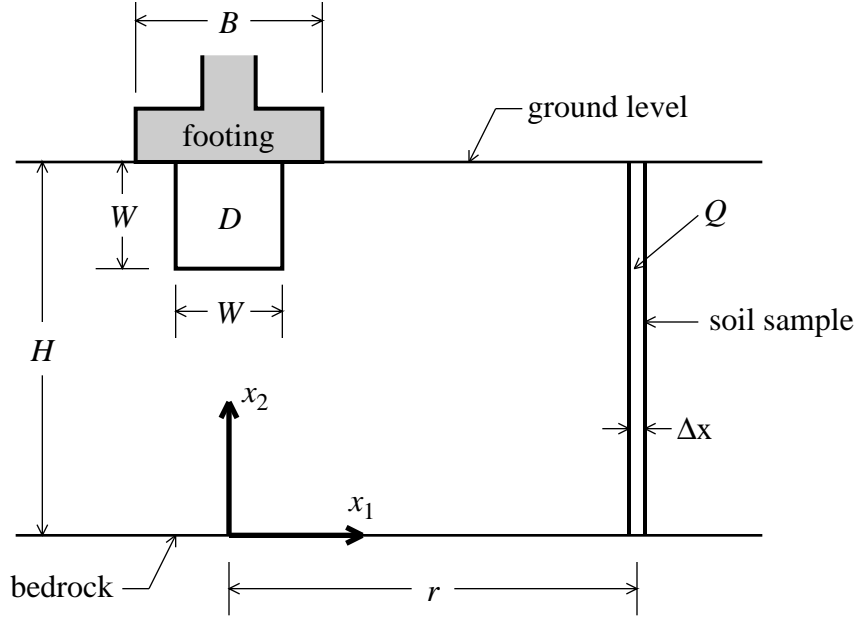


Figure 5. Averaging regions used to predict probability of bearing capacity failure.

Armed with the above information and assumptions, the components of Eq's 26 and 27 can be computed as follows (given the basic statistical parameters of the loads, c , ϕ , the number and locations of the soil samples, and the averaging domain size D);

- 1) Assuming that the total load L is equal to the sum of the maximum live load, L_{Le} , acting over the lifetime of the structure and the static dead load, L_D , i.e. $L = L_{Le} + L_D$, both of which are random, then

$$[33a] \quad \mu_{\ln L} = \ln(\mu_L) - \frac{1}{2} \ln(1 + v_L^2)$$

$$[33b] \quad \sigma_{\ln L}^2 = \ln(1 + v_L^2)$$

where $\mu_L = \mu_{Le} + \mu_D$ is the sum of the mean (max lifetime) live and (static) dead loads, and v_L is the coefficient of variation of the total load defined by

$$[34] \quad v_L^2 = \frac{\sigma_{Le}^2 + \sigma_D^2}{(\mu_{Le} + \mu_D)^2}$$

- 2) With reference to Eq. 3,

$$[35] \quad \mu_{\ln \hat{c}} = \mathbf{E} \left[\frac{1}{m} \sum_{i=1}^m \ln c_i^o \right] = \mu_{\ln c}$$

$$[36] \quad \sigma_{\ln \hat{c}}^2 \simeq \frac{\sigma_{\ln c}^2}{m^2} \sum_{i=1}^m \sum_{j=1}^m \rho(\underline{x}_i^o - \underline{x}_j^o)$$

where \underline{x}_i^o is the spatial location of the center of the i 'th soil sample ($i = 1, 2, \dots, m$) and ρ is the correlation function defined by Eq. 12. The approximation in the variance arises because

correlation coefficients between the local averages associated with observations (in that all tests are performed on samples of some finite volume) are approximated by correlation coefficients between the local average centers. Assuming that $\ln \hat{c}$ actually represents a local average of $\ln c$ over a domain of size $\Delta x \times H$, where Δx is the horizontal dimension of the soil sample, which, for example, can be thought of as the horizontal zone of influence of a CPT or SPT sounding, and H is the depth over which the samples are taken, then $\sigma_{\ln \hat{c}}^2$ is probably more accurately computed as

$$[37] \quad \sigma_{\ln \hat{c}}^2 = \sigma_{\ln c}^2 \gamma(\Delta x, H)$$

3) With reference to Eq. 28,

$$[38] \quad \mu_{\ln \bar{c}} = \text{E} \left[\frac{1}{D} \int_D \ln c(\underline{x}) d\underline{x} \right] = \mu_{\ln c}$$

$$[39] \quad \sigma_{\ln \bar{c}}^2 = \sigma_{\ln c}^2 \gamma(D)$$

where $\gamma(D) = \gamma(W, W)$, as discussed above, is defined by Eq. 13.

4) Since $\mu_{\hat{\phi}} = \mu_{\phi}$ (see Eq. 4), the mean and variance of \hat{N}_c can be obtained using first order approximations to expectations of Eq. 9 (Fenton and Griffiths, 2003), as follows,

$$[40] \quad \mu_{\ln \hat{N}_c} = \mu_{\ln N_c} \simeq \ln \frac{e^{\pi \tan \mu_{\phi}} \tan^2 \left(\frac{\pi}{4} + \frac{\mu_{\phi}}{2} \right) - 1}{\tan \mu_{\phi}}$$

$$[41] \quad \sigma_{\ln \hat{N}_c}^2 \simeq \sigma_{\hat{\phi}}^2 \left(\left. \frac{d \ln \hat{N}_c}{d \hat{\phi}} \right|_{\mu_{\phi}} \right)^2 = \sigma_{\hat{\phi}}^2 \left[\frac{bd}{bd^2 - 1} \left[\pi(1 + a^2)d + 1 + d^2 \right] - \frac{1 + a^2}{a} \right]^2$$

where $a = \tan(\mu_{\phi})$, $b = e^{\pi a}$, $d = \tan \left(\frac{\pi}{4} + \frac{\mu_{\phi}}{2} \right)$. The variance of $\hat{\phi}$ can be obtained by making use of Eq. 17,

$$[42] \quad \begin{aligned} \sigma_{\hat{\phi}}^2 &\simeq \frac{\sigma_{\phi}^2}{m^2} \sum_{i=1}^m \sum_{j=1}^m \rho(\underline{x}_i^o - \underline{x}_j^o) \\ &= \sigma_{\phi}^2 \gamma(\Delta x, H) \end{aligned}$$

where \underline{x}_i^o is the spatial location of the center of the i 'th soil observation ($i = 1, 2, \dots, m$). See Eq. 16 for the definition of σ_{ϕ} .

5) Since $\mu_{\bar{\phi}} = \mu_{\phi}$ (see Eq. 29), the mean and variance of \bar{N}_c can be obtained in the same fashion as for \hat{N}_c (in fact, they only differ due to differing local averaging in the variance calculation). With reference to Eq's 19 and 32

$$[43] \quad \mu_{\ln \bar{N}_c} = \mu_{\ln \hat{N}_c} = \mu_{\ln N_c}$$

$$[44] \quad \sigma_{\ln \bar{N}_c}^2 \simeq \sigma_{\bar{\phi}}^2 \left(\left. \frac{d \ln \bar{N}_c}{d \bar{\phi}} \right|_{\mu_{\phi}} \right)^2 = \sigma_{\bar{\phi}}^2 \left[\frac{bd}{bd^2 - 1} \left[\pi(1 + a^2)d + 1 + d^2 \right] - \frac{1 + a^2}{a} \right]^2$$

$$[45] \quad \sigma_{\bar{\phi}}^2 = \sigma_{\phi}^2 \gamma(D) = \sigma_{\phi}^2 \gamma(W, W)$$

See previous item for definitions of a , b , and d . The variance reduction function, $\gamma(W, W)$ is defined for two dimension by Eq. 13 and Eq. 16 defines σ_ϕ .

- 6) The covariance between the observed cohesion values and the equivalent cohesion beneath the footing is obtained as follows for $D = W \times W$ and $Q = \Delta x \times H$;

$$[46] \quad \text{Cov} [\ln \bar{c}, \ln \hat{c}] \simeq \frac{\sigma_{\ln c}^2}{D^2 Q^2} \int_D \int_Q \rho(x_1 - x_2) d\tilde{x}_1 d\tilde{x}_2 = \sigma_{\ln c}^2 \gamma_{DQ}$$

where γ_{DQ} is the average correlation coefficient between the two areas D and Q . The area D denotes the averaging region below the footing over which equivalent properties are defined and the area Q denotes the region over which soil samples are gathered. These areas are illustrated in Figure 5. In detail, γ_{DQ} is defined by

$$[47] \quad \gamma_{DQ} = \frac{1}{(W^2 \Delta x H)^2} \int_{-W/2}^{W/2} \int_{H-W}^H \int_{r-\Delta x/2}^{r+\Delta x/2} \int_0^H \rho(\xi_1 - x_1, \xi_2 - x_2) d\xi_2 d\xi_1 dx_2 dx_1$$

where r is the horizontal distance between the footing centerline and the centerline of the soil sample column. Eq. 47 can be evaluated by Gaussian quadrature.

- 7) The covariance between \bar{N}_c and \hat{N}_c is similarly approximated by

$$[48] \quad \text{Cov} [\ln \bar{N}_c, \ln \hat{N}_c] \simeq \sigma_{\ln N_c}^2 \gamma_{DQ}$$

$$[49] \quad \sigma_{\ln N_c}^2 \simeq \sigma_\phi^2 \left(\left. \frac{d \ln N_c}{d\phi} \right|_{\mu_\phi} \right)^2 = \sigma_\phi^2 \left[\frac{bd}{bd^2 - 1} \left[\pi(1 + a^2)d + 1 + d^2 \right] - \frac{1 + a^2}{a} \right]^2$$

Substituting these results into Eq's 26 and 27 gives

$$[50] \quad \mu_{\ln Y} = \mu_{\ln L}$$

$$[51] \quad \sigma_{\ln Y}^2 = \sigma_{\ln L}^2 + \left[\sigma_{\ln c}^2 + \sigma_{\ln N_c}^2 \right] \left[\gamma(\Delta x, H) + \gamma(W, W) - 2\gamma_{DQ} \right]$$

which can now be used in Eq. 24 to produce estimates of p_f . If the friction angle is non-random, as in a purely cohesive soil where $\phi = 0$, then $\sigma_{\ln N_c}^2 = 0$ and Eq. 51 simplifies slightly. The primary impact of taking $\phi = 0$ is that the variability of Y is reduced since only one soil parameter is now random. As will be shown shortly, this means that if only cohesion is random, a larger resistance factor can be used, as expected. In other words, the resistance factors presented for the more general $c - \phi$ case are conservative. Letting

$$[52] \quad q = I \left[\alpha_L \hat{L}_L + \alpha_D \hat{L}_D \right]$$

allows the probability of failure to be expressed as

$$[53] \quad p_f = \text{P} \left[Y > q/\phi_g \right] = \text{P} \left[\ln Y > \ln(q/\phi_g) \right] = 1 - \Phi \left(\frac{\ln(q/\phi_g) - \mu_{\ln Y}}{\sigma_{\ln Y}} \right)$$

where Φ is the standard normal cumulative distribution function.

Figure 6 illustrates the best and worst agreement between failure probabilities estimated via simulation and those computed using Eq. 53. The failure probabilities are slightly underestimated at the worst case correlation lengths when the sample location is not directly below the footing. Given all the approximations made in the theory, the agreement is very good (within a 10% relative error), allowing the resistance factors to be computed with confidence even at probability levels which the simulation cannot estimate – the simulation involved only 2000 realizations and so cannot properly resolve probabilities much less than 0.001.

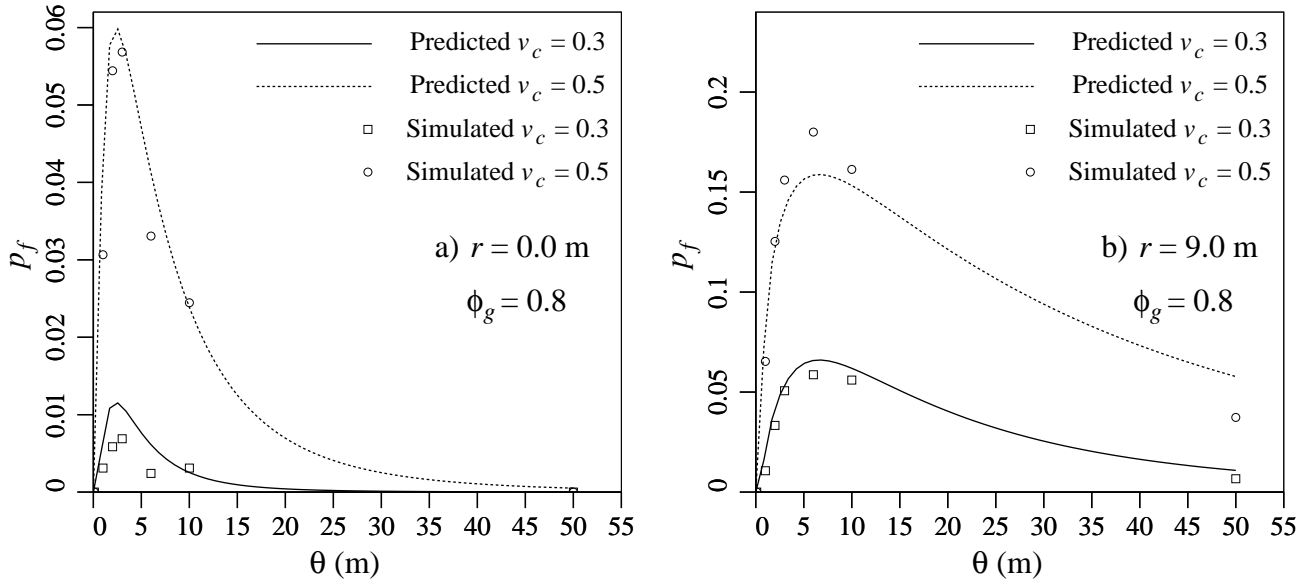


Figure 6. Comparison of failure probabilities estimated from simulation based on 2000 realizations and theoretical estimates using Eq. 53. Plot a) shows probabilities when the soil has been sampled directly under the footing, while b) shows the probabilities when the soil has been sampled 9 m from the footing centerline. Note the change in the vertical scales – the probability of failure is much lower when samples are taken directly under the proposed footing.

4. Required resistance factor

Eq. 53 can be inverted to find a relationship between the acceptable probability of failure, $p_f = p_m$, and the resistance factor, ϕ_g , required to achieve that acceptable failure probability,

$$[54] \quad \phi_g = \frac{I [\alpha_L \hat{L}_L + \alpha_D \hat{L}_D]}{\exp \{ \mu_{\ln Y} + \sigma_{\ln Y} \beta \}}$$

where β is the desired reliability index corresponding to p_m . That is $\Phi(\beta) = 1 - p_m$. For example, if $p_m = 0.001$, then $\beta = 3.09$.

The computation of $\sigma_{\ln Y}$ in Eq. 54 involves knowing the size of the averaging domain, D , under the footing. In turn, D depends on the average mean wedge zone depth (by assumption) under

the footing, which depends on the mean footing width, $\hat{\mu}_B$. Unfortunately, the mean footing width given by Eq. 31 depends on ϕ_g so solving Eq. 54 for ϕ_g is not entirely straightforward. One possibility is to iterate Eq. 54 until a stable solution is obtained. However, the authors have found that Eq. 54 is quite insensitive to the initial size of D and using an ‘average’ value of ϕ_g in Eq. 31 of 0.7 is quite sufficient. In other words, approximating

$$[55] \quad \hat{\mu}_B = \frac{I [\alpha_L \hat{L}_L + \alpha_D \hat{L}_D]}{0.7 \mu_c \mu_{N_c}}$$

allows $\sigma_{\ln Y}$ to be suitably estimated for use in Eq. 54.

In the following, the value of ϕ_g required to achieve three target lifetime failure probability levels (10^{-2} , 10^{-3} , and 10^{-4}) for a specific case (a strip footing founded on a soil with specific statistic parameters) will be investigated. The results are to be viewed relatively. It is well known that the true probability of failure for any design will only be known once an infinite number of replications of that particular design have been observed over infinite time (and thus exposed to all possible loadings). One of the great advantages of probabilistic models is that it is possible to make probabilistic statements immediately, so long as we are willing to accept the fact that the probability estimates are only approximate. In that past history provides a wealth of designs which have been deemed by society to be acceptably reliable (or not, as the case may be), the results presented here need to be viewed *relative* to past designs so that the acceptable risk levels based on the past many years of experience are incorporated. In other words, the results presented in the following, although rational and based on rigorous research, need to be moderated and adjusted by past experience.

The following parameters will be varied in the simulation study to investigate their effects on the resistance factor required to achieve a target lifetime failure probability p_m :

- 1) Three values of p_m are considered, 0.01, 0.001, and 0.0001, corresponding to reliability indices of approximately 2.3, 3.1, and 3.7, respectively.
- 2) The correlation length, θ is varied from 0.0 to 50.0 m.
- 3) The mean cohesion was set to $\mu_c = 100$ kN/m². Four coefficients of variation for cohesion are considered, $v_c = 0.1, 0.2, 0.3,$ and 0.5 . The s factor for the friction angle distribution (see Figure 3) is set correspondingly to $s = 1, 2, 3,$ and 5 . That is, when $v_c = 0.2$, s is set to 2.0, and so on. The friction angle distribution is assumed to range from $\phi_{min} = 0.1745$ radians (10°) to $\phi_{max} = 0.5236$ radians (30°). The corresponding coefficients of variation for friction angle are $v_\phi = 0.07, 0.14, 0.20,$ and 0.29 .
- 4) Three sampling locations are considered: $r = 0, 4.5,$ and 9.0 m from the footing centerline (see Figure 5 for the definition of r).

The design problem considered involves a strip footing supporting loads having means and standard deviations

$$[56a] \quad \mu_{L_e} = 200 \text{ kN/m} \quad \sigma_{L_e} = 60 \text{ kN/m}$$

$$[56b] \quad \mu_D = 600 \text{ kN/m} \quad \sigma_D = 90 \text{ kN/m}$$

Assuming bias factors $k_D = 1.18$ (Becker, 1996) and $k_{L_e} = 1.41$ (Allen, 1975) gives the characteristic loads

$$[57a] \quad \hat{L}_L = 1.41(200) = 282 \text{ kN/m}$$

$$[57b] \quad \hat{L}_D = 1.18(600) = 708 \text{ kN/m}$$

and the total factored design load (assuming $I = 1$) is

$$[58] \quad q = I(\alpha_L \hat{L}_L + \alpha_D \hat{L}_D) = 1.5(282) + 1.25(708) = 1308 \text{ kN/m}$$

So long as the ratio of dead to live load (assumed to be 3.0 in this study), the coefficients of variation of the load (assumed to be $v_{L_e} = 0.3$ and $v_D = 0.15$), and the characteristic bias factors, k_{L_e} and k_D , are unchanged, the results presented here are independent of the load applied to the strip footing. Minor changes in load ratios, coefficients of variation, and bias factors should not result in significant changes to the resistance factor.

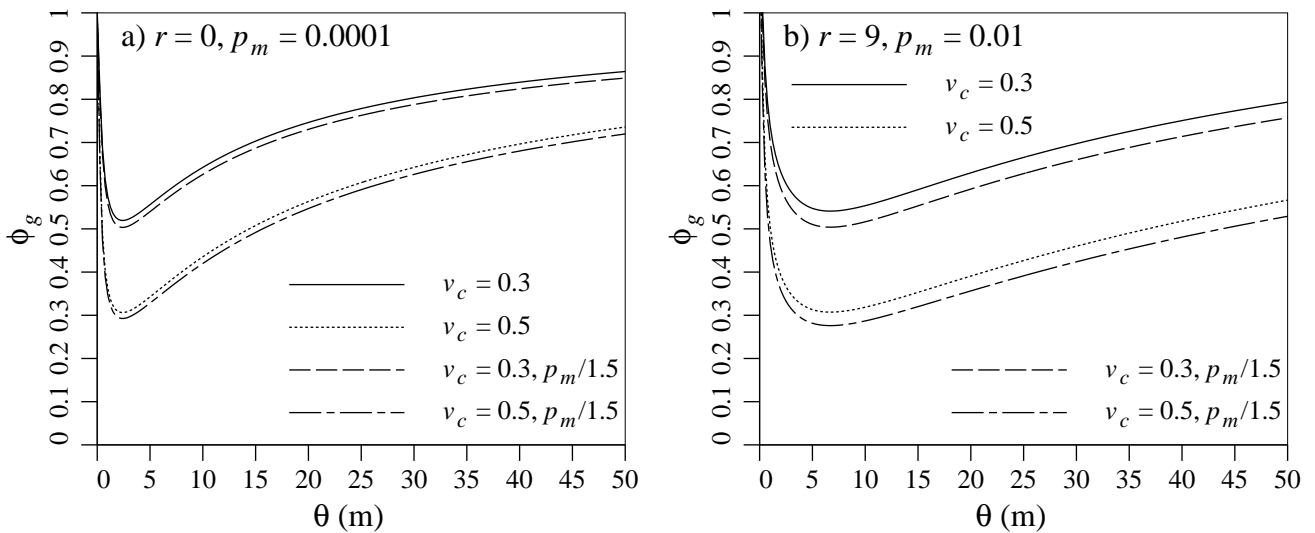


Figure 7. Effect of failure probability underestimation on the resistance factor required by Eq. 54.

Considering the slightly unconservative underestimation of the probability of failure in some cases (see Figure 6b), it is worthwhile investigating how sensitive Eq. 54 is to changes in p_m of the same order as the errors in estimation of p_f . If p_m is replaced by $p_m/1.5$, then this corresponds to underestimating the failure probability by a factor of 1.5, which was well above the maximum difference seen between theory and simulation. It can be seen from Figure 7, which illustrates the effect of errors in the estimation of the failure probability, that the effect on ϕ_g is minor, especially considering all other sources of error in the analysis. Of the cases considered in this study, the ϕ_g values least affected by an underestimation of the probability occur when the soil is sampled under the footing ($r = 0$) and for small p_m , as seen in Figure 7a. The worst case is shown in Figure 7b and all other results (not shown) were seen to lie between these two plots. Even in the worst case of Figure 7b, the change in ϕ_g due to errors in probability estimation are relatively small and will be ignored.

Figures 8, 9, and 10 show the resistance factors required for the cases where the soil is sampled directly under the footing, at a distance of 4.5 m, and at a distance of 9.0 m from the footing centerline, respectively, to achieve the three target failure probabilities. The worst case correlation length is clearly between about 1 to 5 m, as evidenced by the fact that in all plots the lowest resistance

factor occurs when $1 < \theta < 5$ m. This worst case correlation length is of the same magnitude as the mean footing width ($\hat{\mu}_B = 1.26$ m) which can be explained as follows; if the random soil fields are stationary then soil samples yield perfect information, regardless of their location, if the correlation length is either zero (assuming soil sampling involves some local averaging) or infinity. When the information is perfect the probability of a bearing capacity failure goes to zero and $\phi_g \rightarrow 1.0$ (or possibly greater than 1.0 to compensate for the load bias factors). When the correlation length is zero, the soil sample will consist of an infinite number of independent ‘observations’ whose average is equal to the true mean (or true median, if the average is a geometric average). Since the footing also averages the soil properties, the footing ‘sees’ the same true mean (or true median) value predicted by the soil sample. When the correlation length goes to infinity, the soil becomes uniform, having the same value everywhere. In this case, any soil sample also perfectly predicts conditions under the footing.

At intermediate correlation lengths soil samples become imperfect estimators of conditions under the footing, and so the probability of bearing capacity failure increases, or conversely, the required resistance factor decreases. Thus, the minimum required resistance factor will occur at some correlation length between 0 and infinity. The precise value depends on the geometric characteristics of the problem under consideration, such as the footing width, depth to bedrock, length of soil sample, and/or the distance to the sample point. Notice in Figures 8, 9, and 10 that the worst case point does show some increase as the distance to the sample location, r , increases.

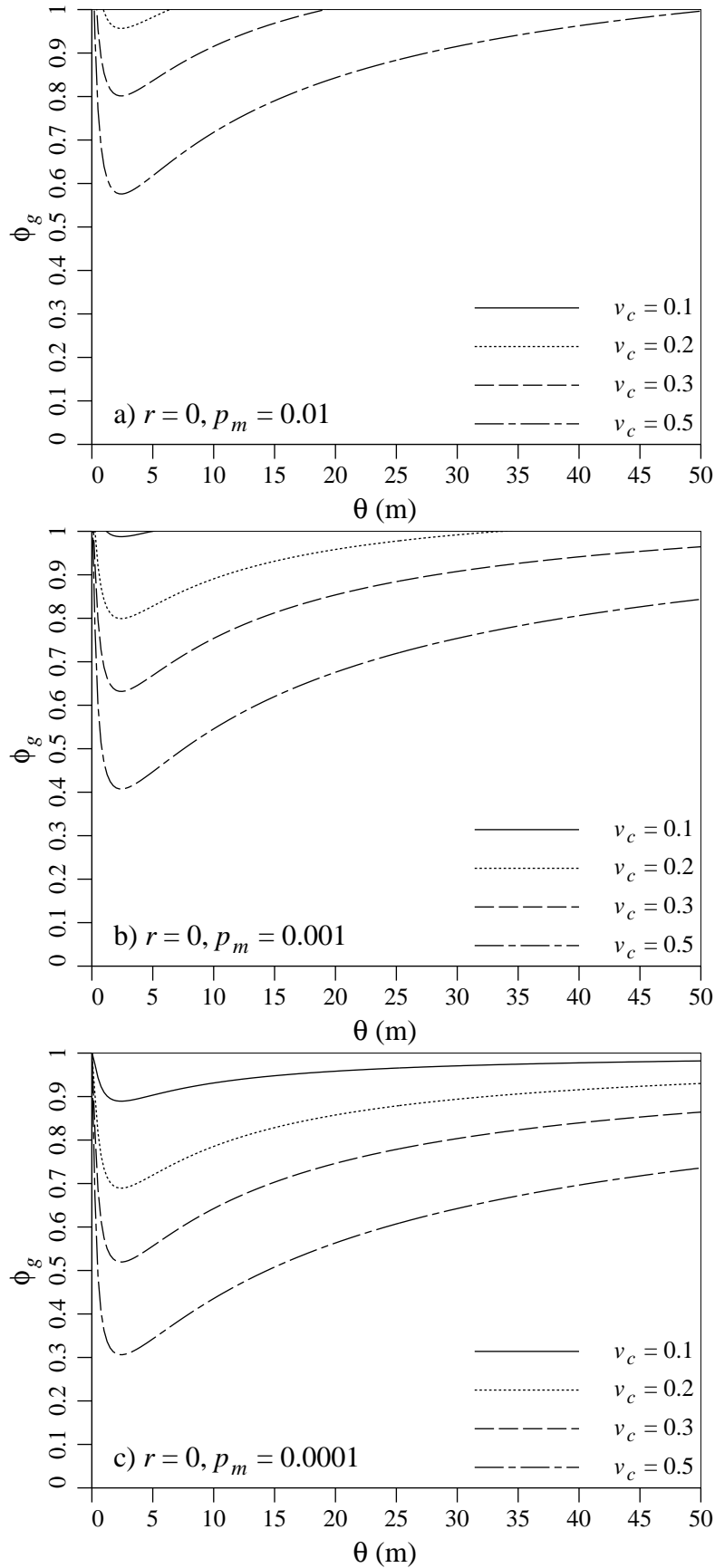


Figure 8. Resistance factors required to achieve acceptable failure probability, p_m , when soil is sampled directly under the footing ($r = 0$).

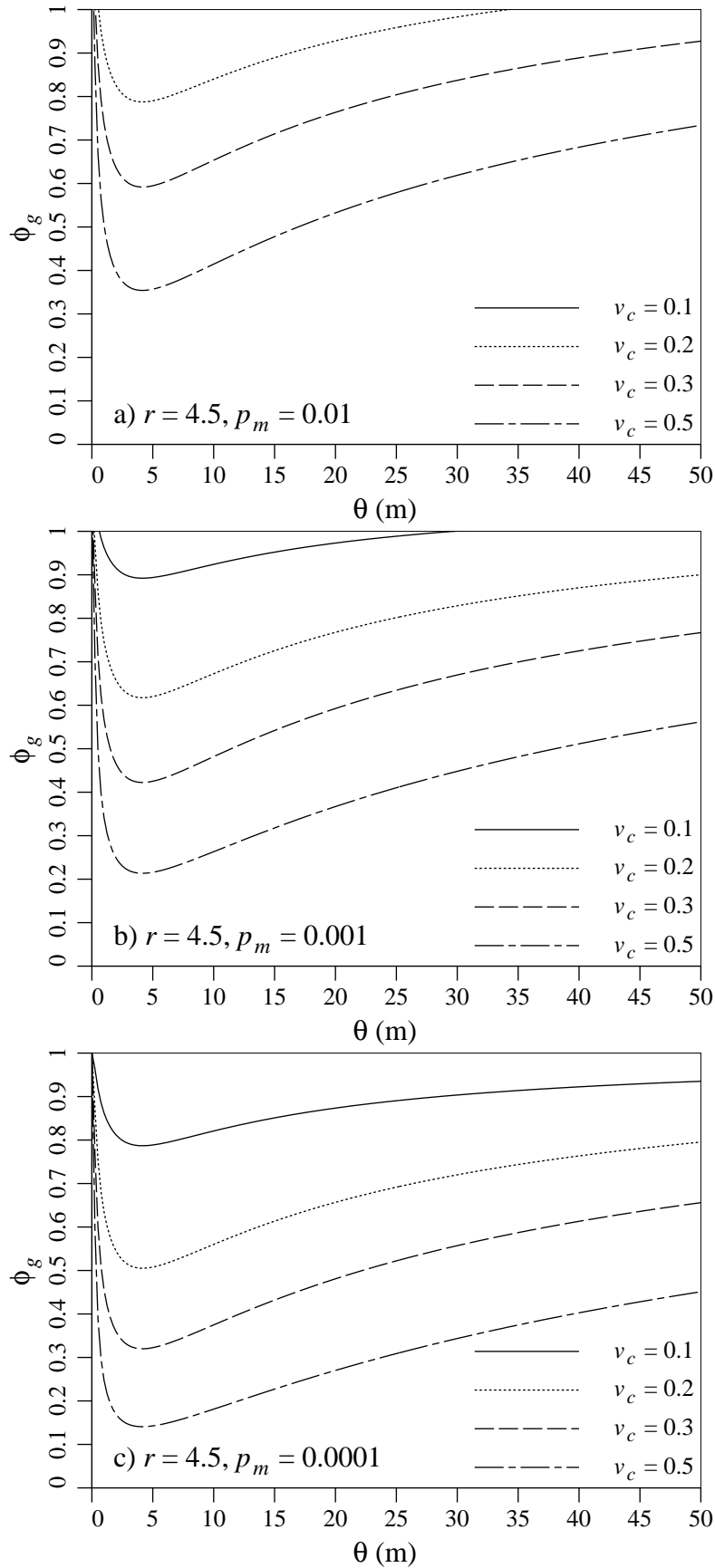


Figure 9. Resistance factors required to achieve acceptable failure probability, p_m , when soil is sampled at $r = 4.5$ m from the footing centerline.

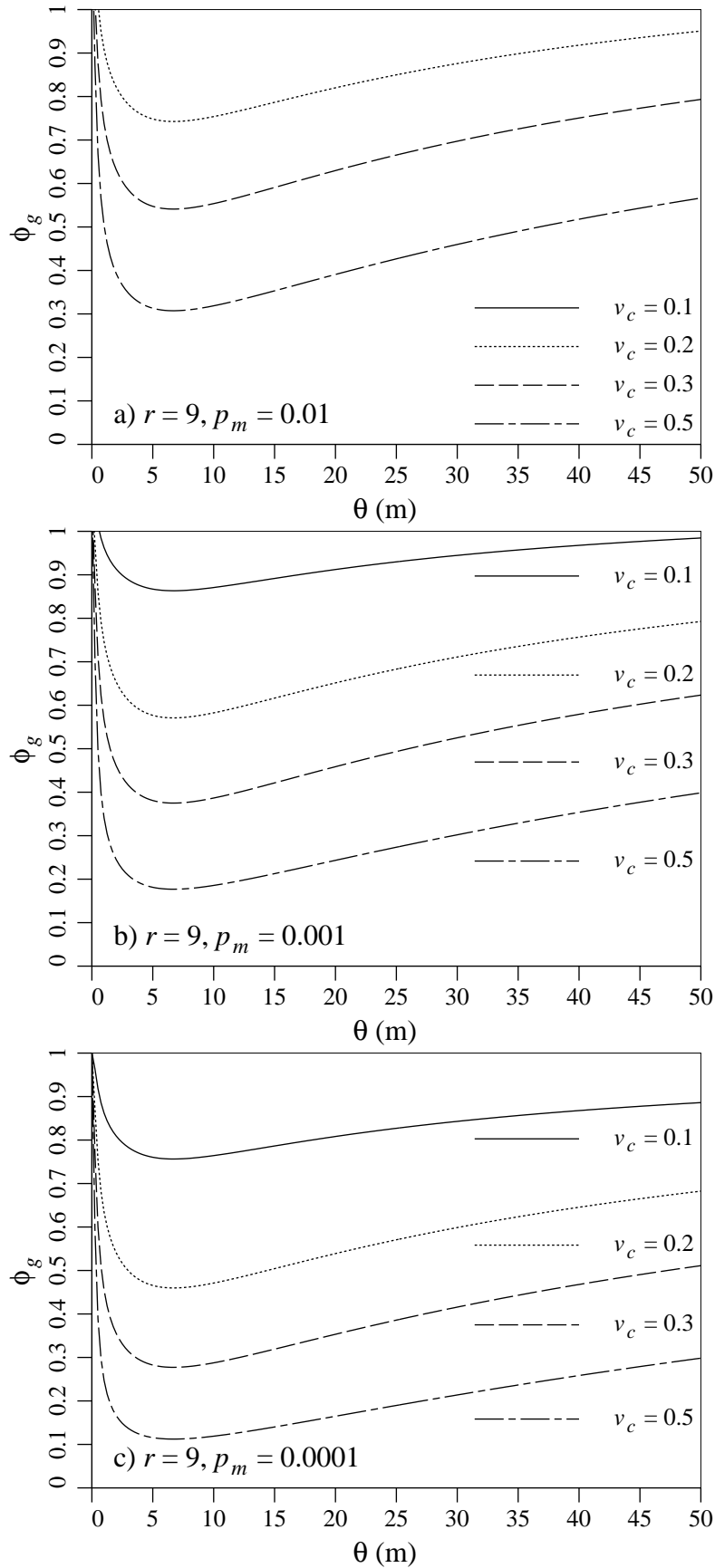


Figure 10. Resistance factors required to achieve acceptable failure probability, p_m , when soil is sampled at $r = 9.0$ m from the footing centerline.

As expected the smallest resistance factors correspond with the smallest acceptable failure probability considered, $p_m = 0.0001$, and with the poorest understanding of the soil properties under the footing (i.e. when the soil is sampled 9 m away from the footing centerline). When the cohesion coefficient of variation is relatively large, $v_c = 0.5$, with corresponding $v_\phi \simeq 0.29$, the worst case values of ϕ_g dip almost down to 0.1 in order to achieve $p_m = 0.0001$. In other words, there will be a significant construction cost penalty if a high reliability footing is designed using a site investigation which is insufficient to reduce the residual variability to less than $v_c = 0.5$.

The simulation results can also be used to verify the theoretically determined resistance factors. This is done by using the simulation-based failure probabilities as values of p_m in the theory and comparing the resistance factor, ϕ_g , used in the simulation to that predicted by Eq. 54. The comparison is shown in Figure 11. For perfect agreement between theory and simulation, the points should align along the diagonal. The agreement is deemed to be very good and much of the discrepancy is due to failure probability estimator error, as discussed next. In general, however, the theory-based estimates of ϕ_g are seen to be conservative, that is somewhat less than seen in the simulations on average.

Those simulations having less than 2 failures out of the 2000 realizations were omitted from the comparison in Figure 11, since the estimator error for such low probabilities is as big, or bigger, than the probability being estimated. In fact, for those simulations having 2 failures out of 2000 (included in Figure 11), the estimated probability of failure is 0.001 which has standard error $\sqrt{0.001(0.999)/2000} = 0.0007$. This error is almost as large as the probability being estimated, having a coefficient of variation of 70%. In fact most of the discrepancies in Figure 11 are easily attributable to estimator error in the simulation. The coefficient of variation of the estimator at the 0.01 probability level is 20%, which is still bigger than most of the relative errors seen in Figure 11 (the maximum relative error in Figure 11 is 0.28 at $\phi_g = 0.5$).

The “worst case” resistance factors required to achieve the indicated maximum acceptable failure probabilities, as seen in Figures 8 through 10, are summarized in Table 2. In the absence of better knowledge about the actual correlation length at the site in question, these factors are the largest values that should be used in the LRFD bearing capacity design of a strip footing founded on a $c - \phi$ soil.

It is noted, however, that the factors listed in Table 2 are sometimes quite conservative. For example, when $v_c = 0.3$, $r = 4.5$ m, and $p_m = 0.001$, Table 2 suggests that $\phi_g = 0.42$ for the $c - \phi$ soil considered here. However, if the soil is undrained, with $\phi = 0$ (all else being the same), then the only source of variability in the shear strength is the cohesion. In this case the above theory predicts a resistance factor of $\phi_g = 0.60$ which is considerably larger than suggested by Table 2.

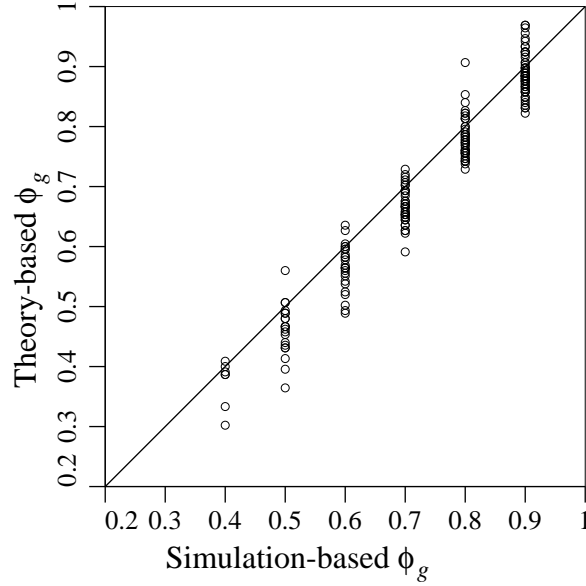


Figure 11. Required resistance factors, ϕ_g , based on simulation versus those based on Eq. 54. For perfect agreement, the points would all lie on the diagonal.

Table 2. Worst case resistance factors for various coefficients of variation, v_c , distance to sampling location, r , and acceptable failure probabilities, p_m .

v_c	$r = 0.0$ m			$r = 4.5$ m			$r = 9.0$ m		
	$p_m = 0.01$	0.001	0.0001	$p_m = 0.01$	0.001	0.0001	$p_m = 0.01$	0.001	0.0001
0.1	1.00	0.99	0.89	1.00	0.89	0.79	1.00	0.86	0.76
0.2	0.96	0.80	0.69	0.79	0.62	0.51	0.74	0.57	0.46
0.3	0.80	0.63	0.52	0.59	0.42	0.32	0.54	0.38	0.28
0.5	0.58	0.41	0.31	0.35	0.21	0.14	0.31	0.18	0.11

To compare the resistance factors recommended in Table 2 to resistance factors recommended in the literature and to current geotechnical LRFD codes, changes in the load factors from code to code need to be taken into account. It will be assumed that all other sources define μ_{L_e} , μ_D , k_{L_e} , and k_D in the same way, which is unfortunately by no means certain. The easiest way to compare resistance factors is to compare the ratio of the resistance factor, ϕ_g , to the total load factor, α . The total load factor, defined for fixed dead to live load ratio, is the single load factor which yields the same result as the individual live and dead load factors, i.e. $\alpha(\hat{L}_L + \hat{L}_D) = \alpha_L \hat{L}_L + \alpha_D \hat{L}_D$. For mean dead to live load ratio $R_{D/L} = \mu_D/\mu_{L_e}$ and characteristic bias factors k_D and k_L ,

$$\begin{aligned}
 [59] \quad \alpha &= \frac{\alpha_L \hat{L}_L + \alpha_D \hat{L}_D}{\hat{L}_L + \hat{L}_D} = \frac{\alpha_L k_L \mu_{L_e} + \alpha_D k_D \mu_D}{k_L \mu_{L_e} + k_D \mu_D} \\
 &= \frac{\alpha_L k_L + \alpha_D k_D R_{D/L}}{k_L + k_D R_{D/L}}
 \end{aligned}$$

which, for $R_{D/L} = 3$, $k_L = 1.41$, $k_D = 1.18$, gives $\alpha = 1.32$. Table 3 compares the ratio of the resistance factors recommended in this study to total load factor ratio with three other sources. The

individual “current study” values correspond to the moderate case where $v_c = 0.3$ and acceptable failure probability $p = 0.001$. The resistance factor derived from the Australian Standard (2004) on bridge foundations assumes a dead to live load ratio of 3.0 (not stated in the code) and that the site investigation is based on CPT tests.

Table 3. Comparison of resistance factors recommended in this study to those recommended by three other sources.

Source	Load Factors	ϕ_g	ϕ_g/α
Current study, $r = 0$ m	$R_{D/L} = 3, \alpha_L = 1.5, \alpha_D = 1.25$	0.63	0.48
Current study, $r = 4.5$ m	$R_{D/L} = 3, \alpha_L = 1.5, \alpha_D = 1.25$	0.42	0.32
Current study, $r = 9.0$ m	$R_{D/L} = 3, \alpha_L = 1.5, \alpha_D = 1.25$	0.38	0.29
Foye et al. (2006)	$R_{D/L} = 4, \alpha_L = 1.6, \alpha_D = 1.20$	0.70	0.54
Canadian Foundation Engrg. Manual (2006)	$R_{D/L} = 3, \alpha_L = 1.5, \alpha_D = 1.25$	0.50	0.38
Australian Standard Bridge Design (2004)	$R_{D/L} = 3, \alpha_L = 1.8, \alpha_D = 1.20$	0.45	0.33

Apparently the resistance factor recommended by Foye et al. (2006) assumes very good site understanding – they specify that the design assumes a CPT investigation which is presumably directly under the footing. Foye’s recommended resistance factor is based on a reliability index of $\beta = 3$, which corresponds to $p_m = 0.0013$, which is very close to that used in Table 3 ($p_m = 0.001$). The small difference between the “current study” $r = 0$ result and Foye’s may be due to differences in load bias factors – these are not specified by Foye et al.

The resistance factor specified by the Canadian Foundation Engineering Manual (Canadian Geotechnical Society, 2006) is somewhere between that predicted here for the $r = 0$ and $r = 4.5$ m results. The CFEM resistance factor apparently presumes a reasonable, but not significant, understanding of the soil properties under the footing (e.g. $r \simeq 3$ m rather than $r = 0$ m). The corroboration of the rigorous theory proposed here by an experience-based code provision is, however, very encouraging. The authors also note that the CFEM is the only source listed in Table 3 for which the live and dead load bias factors used in this study can be reasonably assumed to also apply.

The Australian Standard AS 5100.3 (2004) resistance factor ratio is very close to that predicted here using $r = 4.5$ m. It is probably reasonable to assume that the Australian Standard recommendations correspond to a moderate level of site understanding (e.g. $r = 4.5$ m) and an acceptable failure probability of about 0.001.

5. Summary

One of the main impediments to the practical use of this paper is that it depends on an a-priori knowledge of the variance, and, to a lesser extent since “worst case” results are presented above, the correlation structure of the soil properties. However, assuming that at least one CPT or SPT sounding (or equivalent) is taken in the vicinity of the footing, it is probably reasonable to assume that the residual variability is reduced to a coefficient of variation of no more than about 0.3, and often considerably less (the results collected by other investigators, e.g. Phoon and Kulhawy, 1999, suggest that this may be the case for ‘typical’ site investigations). If this is so, the resistance factors recommended in Table 2 for $v_c = 0.3$ are probably reasonable for the load and bias factors assumed in this study.

The resistance factors recommended in Table 2 are conservative in (at least) the following ways;

- 1) it is unlikely that the correlation length of the residual random process at a site (after removal of any mean or mean trend estimated from the site investigation, assuming there is one) will actually equal the “worst case” correlation length,
- 2) the soil is assumed weightless in this study. The addition of soil weight will reduce the failure probability and so result in higher resistance factors for fixed acceptable failure probability,
- 3) sometimes more than one CPT or SPT sounding is taken at the site in the footing region, so that the site understanding may exceed even the $r = 0$ m case considered here if trends and layering are carefully accounted for,
- 4) c and ϕ are assumed independent, rather than negatively correlated, which leads to a somewhat higher probability of failure and correspondingly lower resistance factor, and so somewhat conservative results. Since the effect of positive or negative correlation of c and ϕ was found by Fenton and Griffiths (2003) to be quite minor, this is not a major source of conservatism.

On the other hand, the resistance factors recommended in Table 2 are unconservative in (at least) the following ways;

- 1) measurement and model errors are not considered in this study. The statistics of measurement errors are very difficult to determine, since the true values need to be known. Similarly, model errors, which relate both the errors associated with translating measured values (e.g. CPT or SPT measurements to friction angle values) and the errors associated with predicting bearing capacity by an equation such as Eq. 5 are quite difficult to estimate simply because the true bearing capacity along with the true soil properties are rarely, if ever, known. In the authors’ opinions this is the major source of unconservatism in the presented theory. When confidence in the measured soil properties or in the model used is low, the results presented here can still be employed by assuming that the soil samples were taken further away from the footing location than they actually were (e.g. if low-quality soil samples are taken directly under the footing, $r = 0$, the resistance factor corresponding to a larger value of r , say $r = 4.5$ m should be used).
- 2) the failure probabilities given by the above theory are slightly underpredicted when soil samples are taken at some distance from the footing. The effect of this underestimation on the recommended resistance factor has been shown to be relatively minor but nevertheless is slightly unconservative.

To some extent the conservative and unconservative factors listed above cancel one another out. Figure 11 suggests that the theory is generally conservative if measurement errors are assumed to

be insignificant. The comparison of resistance factors presented in Table 3 demonstrates that the ‘worst case’ theoretical results presented in Table 2 agree quite well with current literature and LRFD code recommendations, assuming moderate variability and site understanding, suggesting that the theory is reasonably accurate. In any case, the theory provides an analytical basis to extend code provisions beyond mere calibration with the past.

The results presented in this paper are for a $c - \phi$ soil in which both cohesion and friction contribute to the bearing capacity, and thus to the variability of the strength. If it is known that the soil is purely cohesive (e.g. “undrained clay”), then the strength variability comes from one source only. In this case, not only does Eq. 51 simplify since $\sigma_{\ln N_c}^2 = 0$, but because of the loss of one source of variability, the resistance factors increase significantly. The net result is that the resistance factors presented in this paper are conservative when $\phi = 0$. Additional research is needed to investigate how the resistance factors should generally be increased for “undrained clays”.

The effect of anisotropy in the correlation lengths has not been carefully considered in this study. It is known, however, that increasing the horizontal correlation length above the worst case length is conservative when the soil is not sampled directly below the footing. When the soil is sampled directly under the footing, weak spatially extended horizontal layers below the footing will obviously have to be explicitly handled by suitably adjusting the characteristic soil properties used in the design. If this is done, then the resistance factors suggested here should still be conservative. The theory presented in this paper easily accommodates the anisotropic case.

One of the major advantages to a table such as Table 2 is that it provides geotechnical engineers with evidence that increased site investigation will lead to reduced construction costs and/or increased reliability. In other words, Table 2 is further evidence that you pay for a site investigation whether you have one or not (Institution of Civil Engineers, 1991).

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List of Symbols

The following symbols are used in this paper:

- $a = \tan \mu_\phi$
- $A =$ load tributary area
- $b = e^{\pi a}$
- $B =$ strip footing width
- $c =$ cohesion
- $\bar{c} =$ geometric average of cohesion field over domain D
- $\hat{c} =$ geometric average of observed (sampled) cohesion values
- $c_i^o =$ observed (sampled) cohesion value
- $d = \tan \left(\frac{\pi}{4} + \frac{\mu_\phi}{2} \right)$
- $D =$ effective soil property averaging domain centered under footing $= W \times W$
- $D_1 = x_1$ dimension of the averaging domain D
- $D_2 = x_2$ dimension of the averaging domain D
- $E[\cdot] =$ expectation operator
- $G_{\ln c} =$ standard normal random field (log-cohesion)
- $G_\phi =$ standard normal random field (underlying friction angle)
- $H =$ depth to bedrock and depth of assumed soil sample
- $I =$ importance factor
- $k_{L_e} =$ extreme lifetime live load bias factor
- $k_D =$ dead load bias factor
- $L =$ total true (random) footing load, kN/m
- $\hat{L}_i = i$ 'th characteristic load effect
- $L_D =$ true (random) dead load, kN/m
- $\hat{L}_D =$ characteristic dead load $= k_D \mu_D$, kN/m
- $L_{L_e} =$ true (random) maximum live load over design life, kN/m
- $\hat{L}_L =$ characteristic live load $= k_{L_e} \mu_{L_e}$, kN/m
- $m =$ number of soil observations
- $N_c =$ N-factor associated with cohesion, which is a function of ϕ
- $\bar{N}_c =$ effective N-factor associated with cohesion, which is based on an arithmetic average of the friction angle over domain D

\hat{N}_c = characteristic N-factor associated with cohesion, which is based on an arithmetic average of the observed friction angles over domain Q (m soil sample observations)

p_f = probability of bearing capacity failure

p_m = maximum acceptable probability of bearing capacity failure

q_u = ultimate bearing stress

\hat{q}_u = ultimate bearing stress estimated from characteristic soil properties

q = factored design load = $I(\alpha_L \hat{L}_L + \alpha_D \hat{L}_D)$

Q = characteristic soil property averaging domain = $\Delta x \times H$

r = distance between soil sample and footing center, m

$R_{D/L}$ = ratio of mean dead load and mean extreme lifetime live load

\hat{R}_u = ultimate resistance based on characteristic soil properties

R_u = true ultimate resistance (random)

s = scale factor used in distribution of ϕ

v_c = coefficient of variation of cohesion

v_D = coefficient of variation of dead load

v_L = coefficient of variation of total load

v_{L_e} = coefficient of variation of extreme lifetime load

v_ϕ = coefficient of variation of friction angle

W = side dimension of effective averaging domain D

\underline{x} = spatial coordinate, (x_1, x_2) in 2-D

\underline{x}_i^o = spatial coordinate of the center of the i 'th soil sample

x_i = spatial direction (x_1 and x_2) or i 'th soil property observation

Y = true load times ratio of estimated to effective bearing capacity

α = total load factor

α_i = load factor corresponding to the i 'th load effect

α_L = live load factor

α_D = dead load factor

β = reliability index corresponding to acceptable failure probability, p_m

Δx = horizontal dimension of soil samples

ϕ = friction angle (radians unless otherwise stated)

ϕ_g = resistance factor

$\bar{\phi}$ = arithmetic average of ϕ over domain D

$\hat{\phi}$ = arithmetic average of the m observed friction angles

ϕ_{min} = minimum friction angle

ϕ_{max} = maximum friction angle

- ϕ_i^o = i 'th observed friction angle
- Φ = standard normal cumulative distribution function
- $\gamma(D)$ = common variance function giving variance reduction due to averaging over domain D
- $\gamma_{\ln c}(D)$ = variance function giving variance reduction due to averaging log-cohesion over domain D
- $\gamma_\phi(D)$ = variance function giving variance reduction due to averaging G_ϕ over domain D
- γ_{DQ} = average correlation coefficient between domains D and Q
- μ_c = cohesion mean
- $\mu_{\ln c}$ = log-cohesion mean
- $\mu_{\ln \hat{c}}$ = mean of the estimate of log-cohesion based on a geometric average of cohesion observations
- $\mu_{\ln \bar{c}}$ = mean of the effective log-cohesion based on a geometric average of cohesion over domain D
- μ_{N_c} = mean of N_c
- $\mu_{\ln N_c}$ = mean of $\ln N_c$
- $\mu_{\ln \hat{N}_c}$ = mean of $\ln \hat{N}_c$
- $\mu_{\ln \bar{N}_c}$ = mean of $\ln \bar{N}_c$
- μ_D = mean dead load
- μ_L = mean total load on strip footing, kN/m
- μ_{L_e} = mean extreme live load over design life
- $\mu_{\ln L}$ = mean total log-load on strip footing
- μ_ϕ = mean friction angle
- $\mu_{\hat{\phi}}$ = mean of estimated friction angle
- $\mu_{\bar{\phi}}$ = mean of effective friction angle in zone of influence under footing
- $\mu_{\ln Y}$ = mean of $\ln Y$
- $\hat{\mu}_B$ = estimated mean footing width
- θ = correlation length of the random fields
- $\theta_{\ln c}$ = correlation length of the log-cohesion field
- θ_ϕ = correlation length of the G_ϕ field
- $\rho(\tau)$ = common correlation function
- $\rho_{\ln c}(\tau)$ = correlation function giving correlation between two points in the log-cohesion field
- $\rho_\phi(\tau)$ = correlation function giving correlation between two points in the G_ϕ field
- σ_c = cohesion standard deviation
- σ_D = dead load standard deviation
- σ_{L_e} = standard deviation of extreme lifetime live load
- $\sigma_{\ln L}$ = standard deviation of total log-load
- $\sigma_{\ln c}$ = log-cohesion standard deviation

$\sigma_{\ln \bar{c}} =$ standard deviation of $\ln \bar{c}$

$\sigma_{\ln \hat{c}} =$ standard deviation of $\ln \hat{c}$

$\sigma_{\phi} =$ standard deviation of ϕ

$\sigma_{\bar{\phi}} =$ standard deviation of $\bar{\phi}$

$\sigma_{\ln N_c} =$ standard deviation of $\ln N_c$

$\sigma_{\ln \bar{N}_c} =$ standard deviation of $\ln \bar{N}_c$

$\sigma_{\ln \hat{N}_c} =$ standard deviation of $\ln \hat{N}_c$

$\sigma_{\ln Y} =$ standard deviation of $\ln Y$

$\mathcal{T} =$ vector between two points in the soil domain

$\tau_1 =$ horizontal component of the distance between two points in the soil domain

$\tau_2 =$ vertical component of the distance between two points in the soil domain